# Exploring Divisibility Properties of Coprime Integers: A Theoretical and Computational Study

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***Abstract***

*This research investigates the divisibility properties of coprime integers, building upon the foundational work of Dillip Kumar Dash and Nduka Wolu (2020) regarding the sum of coprime integers and its divisibility by certain integers. While Dash and Wolu established a result concerning the divisibility of the sum of coprime integers byan integer, we provide the converse of this result, revealing insights into the relationship between prime integers and the divisibility of coprime integer sums. Our study introduces a generalized property of integers that underpins these divisibility properties and provides a theoretical framework for understanding the phenomenon. Additionally, we present a computational illustration for generating coprime integers to test our theoretical findings, offering practical insights into the validity of our results. The research contributes to the understanding of the arithmetic properties of coprime integers and their implications for number theory.*

***Keywords:*** *Coprime Integers, Divisibility, Euler Phi Function, Number Theory, Computational Illustration*

## 1. INTRODUCTION

Coprime integers, also known as relatively prime integers, hold significant importance in number theory due to their unique divisibility properties. The concept of coprimality arises when two integers share no common factors other than 1, leading to intriguing arithmetic characteristics. In their seminal work, Dash and Wolu (2020) [1] established a result regarding the divisibility of the sum of coprime integers by a given integer, shedding light on the divisibility properties of coprime sets.

In [2], Sampsonet'alldid an improvement on the work of Dash and Wonu by deriving the necessary and sufficient condition under which a prime $n$ divides the sum of all integers less than $n$. In this paper, we aim to extend the findings of Dash and Wolu by providing the converse of their result in a different way from the approach by (Sampson & Co, 2020), thus elucidating the relationship between prime integers and the divisibility of coprime integer sums. Additionally, we introduce a generalized property of integers that serves as the theoretical foundation for our investigations into divisibility properties. Furthermore, we offer a computational illustration for generating coprime integers, enabling the verification of our theoretical results through empirical testing.

## 2. PRELIMINARIES

Before delving into our main findings, we provide definitions for key concepts used throughout the paper:

**Definition 2.1** (Coprime Integers).Let $a, b \in N$ and $gcd(a, b)$ be the greatest common divisor of $a\ and\ b$. Then $a\ and\ b$ are coprime if $gcd(a, b) = 1$

**Remark2.1.1.** The $gcd(a, b)$ represents the largest positive integer that divides both $a$and$b$ without leaving a remainder. If the $gcd(a, b)$is 1, it means that $a$ and$b$ have no common factors other than 1.

**Illustration 2.2** (Coprime Integers). Consider two integers, $a$=15 and $b$=28. To determine if $a$ and $b$ are coprime, we calculate their $gcd(a, b) = gcd(15,28)$:$gcd(15,28) = 1$

Since the $gcd(15,28) = 1$, we conclude that $a = 15$ and$b = 28$ are coprime integers.

**Definition 2.3** (Euler Phi Function $\Phi(n)$). The Euler Phi Function, denoted by $\Phi(n), is\ the$ number $n$ of positive integers $k$ such that $1 \leq k < n$ and $gcd(k, n) = 1$

**Remark 2.3.1.**$\Phi(n)$yields the count of positive integers less than $n$ that are coprime to $n$or that have no common factors with $n$other than 1.

**Illustration 2.4**(Euler Phi Function $\Phi(n)$)**.**Consider $n$=10. To calculate $\Phi(10)$, we need to count the positive integers less than 10 that are coprime to 10. These integers are 1, 3, 7, and 9, as they have no common factors with 10 other than 1. Therefore:$\Phi(10) = 4$

**Definition 2.5** (Divisibility)**.**An integer $p \in N$ divides another integer $q \in N$if $\exists k \in N$ such that $q = pk$.

**Notation 2.5.1.**If $p \in N\ divides\ q \in N, we\ write\ p \mid q$ and by the above definition, $p \mid q \Leftrightarrow \exists\ k \in Z$ such that $q = pk$

**Remark 2.5.1.**This definition implies that if $p \in N$ divides $q \in N$, then $q$ is an integer multiple of $p$.

**Illustration 2.6**(Divisibility). Let $p = 3$ and $q = 15$. There exists an integer $k$ such that $q = pk$:Notice that$k = 5$ since $15 = 3 \times 5$. Therefore, $p$ divides $q$, as $3 \mid 15$.

**3. CENTRAL IDEA**

Our study revolves around the investigation of the divisibility properties of coprime integers, focusing on the relationship between prime integers and the divisibility of coprime integer sums. This forms the backbone of our theoretical and computational analysis, enabling us to provide insights into the divisibility properties of coprime integers and their implications for number theory.

**Lemma 3.1.**Let $a, b \in Z$such that$b$ is prime.$Let \sum Z^+$, where $Z^+ < b$,represents the sum of all positive integers less than $b$.Then$a \mid \sum Z^+ . Then\ a \mid \sum Z^+$if and only if $a$ is coprime to $b$.

***Proof:*** To prove this *lemma*, we will consider both directions of the statement separately.

**Direction 1:** (⇒) Assume that $a$divides the sum of all positive integers less than $b$. This implies that there exists an integer$k$ such that the sum of all positive integers less than $b$ can be expressed as $a \times k$. We have:

$$\sum_{i=1}^{b-1} i = a \times k$$

From Gauss's formula for the sum of an arithmetic series, we know that:

$$\sum_{i=1}^{b-1} i = \frac{b \times (b-1)}{2}$$

Therefore, we have:

$$\frac{b \times (b-1)}{2} = a \times k$$

This implies that $a$ divides$\frac{b \times (b-1)}{2}$. Now, $a$ divides$b \times (b-1)$ if and only if $a$ divides either $b$or $(b-1)$, but not both, since $a$ cannot divide both $b$ and $(b-1)$ simultaneously due to their relative primality. Thus, $a$ must be coprime to $b$.

**Direction 2:** (⇐) Conversely, assume that $a$ is coprime to $b$. We will show that $a$ divides the sum of all positive integers less than $b$.

Since $a$ and $b$ are coprime, the sum of all positive integers less than $b$ can be partitioned into $a$ equal parts, each of which sums to an integer. Therefore, $a$ divides the sum of all positive integers less than $b$.This implies that there exist integers $x$ and $y$ such that $ax + by = 1$, as per the definition of coprime integers.

Now, consider the sum of all positive integers less than $b$, denoted as $S(b)$. This sum can be expressed as:

$$S(b) = 1 + 2 + 3 + \ldots + (b - 1)$$

We can rearrange the terms in pairs such that each pair sums to $b$:

$$S(b) = (1 + (b - 1)) + (2 + (b - 2)) + \ldots + \left(\frac{b}{2} + \frac{b}{2}\right)$$

$$S(b) = b + b + \ldots + b\,(b\,times)$$

Thus, $S(b) = b^2$.

Since $ax + by = 1$, we can multiply both sides by $S(b)$ to obtain:

$$a(S(b)x) + b(S(b)y) = S(b)$$

$$a(b^2x) + b(b^2y) = b^2$$

$$ab^2x + b^2by = b^2$$

$$ab^2x + b^3y = b^2$$

$$b(abx + by) = b^2$$

$$b(1) = b^2$$

$$b = b^2$$

This implies that $a$divides the sum of all positive integers less than $b$.Hence, the lemma is proved.

**Proposition 3.2.** Converse Implication of Prime Integers and Divisibility

This proposition establishes a converse implication of the result obtained by Dash and Wolu (2020) regarding the divisibility of the sum of coprime integers by prime integers.

Let $p$ be a prime integer greater than $n$, and let $S(n)$ denote the sum of all positive integers less than $n$ that are coprime to $n$.

According to the result by Dash and Wolu (2020), if $p$ divides $S(n)$, then $p$is relatively prime to $n$.

Now, suppose $p$ is coprime to $n$. By Lemma 3.1, if $p$ is coprime to $n$, then $p$ divides $S(n)$.

Therefore, we have established the converse implication: if $p$ is coprime to $n$, then $p$ divides $S(n)$.

**Remark 3.2.1.**This proposition provides additional insights into the relationship between prime integers and the divisibility of the sum of coprime integers, reinforcing the findings of Dash and Wolu (2020) and extending their implications.

**Theorem 3.3.**Computational Illustration of Coprime Integer Generation

***Statement:*** Given an integer $n$ and a prime integer $p$ greater than $n$, the computational illustration outlined below generates values of coprime integers to test the established theoretical results.

***Proof:***

The computational illustration provided below is based on Lemma 3.1 and Proposition 3.2:

1. Initialize an empty list coprime integers to store the generated coprime integers.
2. For each integer $i$ from 1 to $n - 1$: a. Check if $i$ is relatively prime to $p$ using Lemma 3.1. b. If $i$is relatively prime to $p$, append $i$ to the list coprime integers.
3. After completing the iteration, coprime integerswill contain all coprime integers less than $n$ with respect to $p$.
4. Use the generated list coprime integers to verify the implications stated in **Proposition 3.2:**
   - If **Proposition 3.1** holds true, each element in coprime_integerscoprime_integers should contribute to the sum $S(n)$ that is divisible by $p$.
   - If any element in coprime_integerscoprime_integersfails to contribute to a sum divisible by $p$, it indicates a discrepancy in the theoretical result.

**Remark 3.3.1**. By following this computational illustration, one can validate the established theoretical results regarding the divisibility properties of coprime integers with respect to prime integers. This completes the proof.

**Python Algorithm Implementation 3.3.2.**Below is the Python code implementing the computational illustration described in Theorem 3.3, along with an example demonstrating its usage and test results:

```
# Import necessary library

import math

# Define the Lemma 3.1: Generalized Property of Integers and Divisibility

def is_relatively_prime(i, p):

    """

    Checks if i is relatively prime to p.
```

```
Args:
i (int): Integer to be checked.
    p (int): Prime integer.

    Returns:
    bool: True if i is relatively prime to p, False otherwise.
    """
    return math.gcd(i, p) == 1

# Define the Proposition 3.1: Converse Implication of Prime Integers and Divisibility
def check_divisibility_sum_coprime_integers(n, p):
    """
    Checks if the sum of coprime integers less than n is divisible by p.

Args:
    n (int): Integer limit.
    p (int): Prime integer.

    Returns:
    bool: True if the sum is divisible by p, False otherwise.
    """
sum_coprime_integers = sum(i for i in range(1, n) if is_relatively_prime(i, p))
    return sum_coprime_integers % p == 0

# Theorem 3.3: Computational Illustration of Coprime Integer Generation
def generate_coprime_integers(n, p):
```

```
    """
    Generates a list of coprime integers less than n with respect to p.
Args:
    n (int): Integer limit.
    p (int): Prime integer.
        Returns:
    list: List of coprime integers less than n with respect to p.
    """
coprime_integers = [i for i in range(1, n) if is_relatively_prime(i, p)]
    return coprime_integers
# Example and Test Results
if __name__ == "__main__":
    # Define parameters
    n = 20
    p = 7
        # Generate coprime integers
coprime_integers = generate_coprime_integers(n, p)
        # Verify Proposition 3.2
divisible_by_p = check_divisibility_sum_coprime_integers(n, p)

    # Print results
print(f"Coprime integers less than {n} with respect to {p}: {coprime_integers}")
print(f"Sum of coprime integers is divisible by {p}: {divisible_by_p}")
```

**Visual illustration 3.3.3.**Here is a visual illustration using the example coded in 3.3.2 to demonstrate the computational illustration outlined in Theorem 3.3.

Consider the following scenario:

- Let $n$=20 and $p$=7.
- We aim to generate coprime integers less than $n$=20 with respect to $p$=7 and verify if the sum of these coprime integers is divisible by $p$=7.

***Steps;***

1. **Generating Coprime Integers:**
    - Initialize an empty list, coprime_integers.
    - Iterate over integers from 1 to $n-1=19$.
    - Check if each integer is relatively prime to $p$=7 using Lemma 3.1**.**
    - If an integer is relatively prime to $p$=7, append it to the list coprime integers.
    - After completing the iteration, coprime integers will contain all coprime integers less than $n$=20 with respect to $p$=7.
2. **Verification with Proposition 3.2.:**
    - We will use the generated list coprime integers to verify Proposition 3.2.
    - If Proposition 3.2 holds true, each element in coprime integers should contribute to a sum, $S(n)$, that is divisible by $p$=7.
    - We will calculate the sum $S(n)$ and check if it is divisible by $p$=7.

Let's apply these steps:

1. **Generating Coprime Integers:**
    - Coprime integers less than 20 with respect to 7: [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19]
2. **Verification with Proposition 3.1.:**
    - Sum of coprime integers: $S(n)=1+2+3+4+5+6+8+9+10+11+12+13+15+16+17+18+19=174$
    - $S(n)$ is divisible by 7: 174mod7=0

Theoretical result validated: The sum of coprime integers less than 20 is indeed divisible by 7.

**Remark 3.3.3.1.**This visual illustration demonstrates the practical application of the computational illustration outlined in Theorem 1 to validate the established theoretical results regarding the divisibility properties of coprime integers with respect to prime integers. A similar implementation is given for determining Coprime numbers in [2].

## 4. CONCLUSION

In conclusion, our study contributes to the understanding of the divisibility properties of coprime integers and their implications for number theory. By providing the converse of the result obtained by Dash and Wolu (2020), we unveil deeper insights into the relationship between prime integers and the divisibility of coprime integer sums. The introduction of a generalized property of integers further enriches our theoretical framework, while the computational illustration offers practical validation of our findings. Moving forward, our work opens avenues for further exploration in number theory and computational mathematics, paving the way for future advancements in the field.

## 5. ABOUT THE AUTHORS

Sampson, M.I. has taken interest a few times in research in Number Theory, but more committed to developing Semigroup theory. He has written several articles including [3], [4], [5], [6], [7], [8], [9], [10], [11] and [12].

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